Objective mathematics in a finite unbounded universe

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Abstract

Most mathematicians think that first order arithmetic is objectively true in some sense. Stronger formal systems lead to increasing skepticism. Substituting the philosophical question of which mathematical statements are objective with the question of which mathematical statements are logically determined by events that could in theory occur in the physical universe, as we understand it, provides a partially mathematical definition of one form of objectivity. Because of incompleteness any correct mathematical definition of 'logically determined' can be expanded. Much countable mathematics including the minimal standard model for set theory may meet this definition of objective.

Cantor's uncountability proof and the Löwenheim-Skolem theorem prove that any consistent sufficiently strong first order theory can be expanded with more reals. The absolutely uncountable cardinal hierarchies cannot meet this definition of objective, but they implicitly define tools for the expansion of objective mathematics.

Because objective mathematics is about what may be meaningful in the physical universe, it suggests techniques for constructing partial computer models of mathematical universes that can then be explored experimentally to help develop mathematical understanding and intuition.

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1 Introduction

Most mathematicians think that first order arithmetic is objectively true in some sense. Stronger formal systems lead to increasing skepticism. Paul J. Cohen believed that mathematics beyond the integers is a formalist game[10, 11]. Solomon Feferman wrote "some other philosophy [than Platonism] grounded in inter-subjective human conceptions will have to be sought to explain the apparent objectivity of mathematics[13]."

Pluralism is the view that at some point the foundations of mathematics no longer refer to a single objective truth[20]. Instead there is a choice like that between Euclidean and non Euclidean geometries[19]. The *Platonic multiverse* is a from of pluralism. It claims "that there are many distinct concepts of set, each instantiated in a corresponding set-theoretic universe"[16]. The continuum hypothesis might be true in some of the universes and false in others.

Substituting the philosophical question of which mathematical statements are objective with the question of which mathematical statements are logically determined by events that could in theory occur in the physical universe, as we understand it, provides a partially mathematical definition of one form of objectivity. Because of incompleteness any correct mathematical definition of 'logically determined' can be expanded.

Current physics implies that a finite region of space-time containing finite energy can be fully described within the finite Bekenstein bound on the required number of bits[1]. Mathematics that is determined by events all of which could occur in a fixed finite universe is limited to finite combinatorial statements. However, if time, space and the total information content of the universe is always finite but unbounded and expanding over time[1]. then

[1]The most optimistic model has the universe not only expanding but increasing in mass to create a growing dynamically stable model into the unbounded future. This may or may not be the case but it is a plausible model for the indefinite future given the vastness of the universe we inhabit.

statements could be logically determined by a recursively enumerable sequence[2] of objective events *each* of which could occur within the physical universe[3]. Possible objective physical event is synonymous with finite combinatorial statement in the context of this paper.

Logically determined can be generalized by the assumption that statements logically determined by a recursively enumerable sequence of objective statements are objective. All finite combinatorial statements are objective and everything is built up from them. The finite combinatorial statements are structured in the form of a tree. The relationship between statements may be complex, indirect and involve infinite subsections of the tree. All statements in the tree are recursively enumerable by a nondeterministic computer[4] that enumerates all the statements at every node in the tree.

This returns to the old idea that infinity is a potential that can never be fully realized. Throughout the remainder of this paper objective without a qualifier means built up from events that could occur in an unbounded, but always finite universe[5] but including relationships involving a countably infinite number of these events.

1.1 Uncountable sets

The uncountable cardinal hierarchy can play at least two roles in objective mathematics. Every set in the minimal standard model for Zermelo Frankel set theory (**ZF**) (assuming **ZF** has a model) has a definition and is constructible from more primitive sets down to the empty set. As seen from outside, this model is countable. It is possible that all statements in **ZF** about this model are logically determined by a recursively enumerable sequence of objective statements. The iterative definition of the minimal standard model starts with objective sets (the integers and the set of all integers) and maps objective sets to objective sets as described in Section 4. I suspect that the ordinal required to iterate this construction to obtain the minimal model for **ZF** is objective and the same is true of **ZF** plus some large cardinal axioms. While the unconstrained versions of these theories cannot be objective, they may have an objective minimal model.

Combining Cantor's uncountability proof with the Löwenheim-Skolem[6] theorem leads to an incompleteness result. No first order formulation of mathematics can define all reals. The

[2]The events must be recursively enumerable to be precisely defined.

[3]Current cosmology suggests there are time and size limits to the evolution of structure, but they are very large and highly speculative bounds. Almost nothing is known about dark matter and dark energy that appears to make up over 95% of the mass of the universe. Cosmology may be wrong about the fate of the universe. Its predictions have changed dramatically with more data and a deeper understanding.

[4]A nondeterministic computer can simulate a recursively enumerable set of computer programs by switching between them so eventually every program is fully executed. In the process of doing so it may run across additional programs that must be executed.

[5]Although I want to emphasize the mathematics relevant to physical reality as we understand it I do have philosophical arguments that define the boundaries of objective mathematics. These views center on the word exists. I think structure is how we view objects from the outside and consciousness is what it feels like for particular structures to exist. I have proposed the Totality Axiom: *Immediate experience in some form is the essence and totality of the existence of physical structure and structure is the only aspect of existence that can be communicated*[2]. To exist is to be aware in some sense and that awareness cannot be infinite because it is definite. Unlike infinite sets, if you add to it you *always* change it.

[6]The Löwenheim-Skolem theorem implies that any formal first order system that has a model must have a countable model.

uncountable cardinals can be seen as a hierarchy of incompleteness results that can be used iteratively to define more reals and other countable mathematical structures. An example of this is the use of uncountable ordinals to define large recursive ordinals using ordinal collapsing (see Section 3.5).

1.2 Computer based mathematical models

Computer based mathematical models are rare in the foundations of mathematics. Computers play a role in deriving and more commonly verifying mathematical proofs. However most of **ZF** is considered beyond computer modelling. A shift in focus in the context of objective mathematics can change this for some and perhaps most of countable mathematics. Computers can directly operate on recursive ordinals and many sets constructible from them. For other structures two approaches are useful.

The first uses recursive functions operating on nonrecursive domains as opposed to Turing machine[7] (TM) oracles[8]. Kleene's notations for recursive ordinals, $\mathcal{O}$[18, 5] is an example. A TM with an oracle for $\mathcal{O}$ can only be simulated with a partial oracle for known members or nonmembers of $\mathcal{O}$. When the TM queries an integer which is not known to be a member or nonmember the computation cannot continue. In contrast a recursive process designed to operate on *all* members of $\mathcal{O}$ can be applied to examples known to be in $\mathcal{O}$. Such limited computer experiments may be helpful in expanding the recursive and countable ordinal hierarchies and in gaining intuition about related mathematics.

The other approach concerns the type of structures in the model. Kleene's $\mathcal{O}$ contains notations for *all* recursive ordinals (those $< \omega_1^{\mathrm{CK}}$). However the members of $\mathcal{O}$ are not recursively enumerable. Recursively recognized ordinal notations are usually required to do computer modelling of the ordinals. A practical system cannot usually include notations for all recursive ordinals. There will be gaps including at least a gap starting at the limit of the recursive ordinals recognized in the system and ending at ω_1^{CK}.

The difference between complete and recursive models is illustrated by two different approaches for defining ordinal notations. Both of these use recursive functions operating on nonrecursive domains. The first is a complete model that extends Kleene's $\mathcal{O}$ to larger countable ordinals[5]. In contrast a fully recursive model is constructed for ordinal notations some of which represent ordinals $\geq \omega_1^{\mathrm{CK}}$. This model with gaps is used to implement an interactive ordinal calculator[6].

2 Syntax and semantics

Mathematics combines syntax and semantics in axioms, the laws of logic and the application of the laws to the axioms. The semantics are the abstract structures and relationships

[7] A TM is an idealized computer that can run error free forever. It uses a potentially infinite tape to read from and write to. A universal Turing Machine (UTM) has a simple set of states and associated actions. With these it can, in theory, perform any computation any computer is capable of. A UTM must have a program on its tape when it starts which instructs it to perform a specific computation.

[8] A TM oracle is an imagined device connected to a TM that can be interrogated about whether a given integer is or is not in a specific set. The oracle *always* answers yes or no correctly in a finite time. For example one might imagine an oracle that solves the computer halting problem.

between them that can be proven by logical transformations of the axioms. The definitions (and thus the semantics) of uncountable sets in first order theories are necessarily incomplete and expandable. If **ZF** has any model it has many alternative models. Thus pluralism is a logical possibility. Forcing[9] can add sets to form new models with engineered properties. One way to avoid the ambiguity is through the minimal model of a theory that excludes all sets not explicitly defined. A related way is to add semantic constraints to the laws of logic.

A poof in objective mathematics consists of a conventional proof plus a proof that the result is objectively determined. A partial formalization of objective mathematics that does not require a proof of objectivity is available[4]. However some well founded structures that require limited quantifiers over the reals are objectively determined and difficult to specify with formalization alone. Table 1 gives a hierarchy of definitions of logically determined. Definitions that require limited quantification over the reals start at item 4.

A guiding principal in defining the semantics of objective statements and sets is to ask what questions may be directly relevant to beings that inhabit an unbounded finite recursive universe. For example one might want to know if the human species will have an infinite chain of descendant species. Given an unbounded universe with recursive laws of physics and a recursive way to identify new species, this is a statement of the form $\exists_{r\in\mathbb{R}}\forall_{n\in\mathbb{N}}\ c(r,n)$ where $c(r,n)$ says that a finite chain of descendant species of length n encoded by an initial sequence in r will exist. This question is objective because it refers to an unbounded recursively enumerable sequence of possible physical events that are meaningful to finite beings.

2.1 Objective sets

The empty set and all sets constructed from it and other objective sets using only the axioms of union and unordered pairs[9] are objective. Only finite and countably infinite sets can be objective. A countably infinite set is objective if its members are defined by a recursive enumerable sequence of objective set definitions. These definitions can be of finite sets of objective sets and those of the form $\{x|x \in y \land p(x)\}$ where y is an objective set and $p(x)$ is an objective statement for all $x \in y$. For example the set of all integers, ω, is objective. Its semantics is a recursive process that enumerates the integers. More methods for defining objective infinite sets are in sections 3 and 4.

In defining an infinite set, the enumeration of members may contain duplicates. Thus not every element of the enumeration necessarily defines a new member.

2.2 Objective statements

An objective statement is defined to be either a finite combinatorial statement or a recursively enumerable sequence of objective statements and a relationship between the enumerated statements such as the infinite AND or OR of all of them[9]. Table 1 summarizes initial levels of a hierarchy of mathematically defined classes of objective statements. These are developed in sections 3 and 4.

Following are some elementary properties of objective statements about objective sets.

[9] A statement of the form $\forall_{x\in\mathbb{N}}\ \varphi(x)$ is equivalent to the sequence of statements: $\varphi(0) \land \varphi(1) \land \varphi(2) \land \varphi(3) \land ...\varphi(n) \land ....$ A similar definition applies to $\exists$ and OR.

1. Objectively bound universal and existential quantifiers
   $\forall_{x \in a}\, p(x)$ and $\exists_{x \in a}\, p(x)$ are objective statements if a is an objective set and if, for all $x \in a$, $p(x)$ is an objective statement. $p(x)$ may contain quantifiers as long as it is an objective statement for $x \in a$.

   The recursively enumerable sequence that determines this statement are the infinite AND or OR of the statements $p(x)$ where x takes on all members of a.

2. Equality
   If a and b are objective sets then $a = b$ is an objective statement.

   If both a and b are finite sets than $a = b$ is a finite combinatorial statement. If not the nth elements in the enumerations of a and b are indexed as a_n and b_n. The ordering of a_n and b_n may differ even if $a = b$. Thus a two level tree is required. Each entry in the two level tree has the index $[n, m]$. The top level (indexed by n) is the infinite AND of infinite ORs at the second level (indexed by m). The statement at index $[n, m]$ is $a_n = b_m$. This insures that every element of a is an element of b. To insure that every element of b is an element of a requires a second tree with the roles of a and b reversed. The statement is the logical and of these two trees.

3. Membership
   If a and b are objective sets then $a \in b$ is an objective statement.

   The recursively enumerable sequence that determines this are the statements $a = x$ where x ranges over the enumerated elements of b. The infinite OR of these statements is logically equivalent to $a \in b$.

3 Logically determined

The Church-Turing thesis states that anything that is effectively computable (that is computable by an algorithm) is computable on a UTM (see note 7) by including on the initial tape a program for that algorithm and the data it is to operate on. Because effectively computable and algorithm have no precise mathematical definition, except that offered by the Church-Turing thesis, it is impossible to prove this thesis. Most mathematicians and computer scientists think it is true. This thesis seems to define the limit of universes that one can model with completely predictable experiments[10].

What a computer program does at every state transition is predictable, but if it will *ever* do something such as terminate or halt is not in general predictable. Yet this question is logically determined. There is a hierarchy of such questions. For example a more a general question is will a TM have an infinite number of outputs. Initial levels of this hierarchy are listed in Table 1 and defined in subsections of this section and Section 4.

3.1 Infinite AND ($\forall$) and OR ($\exists$)

$\forall_{n \in \mathbb{N}}\, r(n)$ and $\exists_{n \in \mathbb{N}}\, r(n)$ where $r(n)$ is a recursive relation on the integers is logically determined. See Item 1 in Section 2.2.

[10]The irreducibly random laws of quantum mechanics mean some predictions will only be statistical.

1. finite combinatorial statements
2. infinite AND and OR ($\forall$ and $\exists$) (Section 3.1)
3. $\forall$ and $\exists$ recursive nesting (Section 3.2)
4. well founded trees (Section 3.3)
5. trees with ordinal labeled nodes (Section 3.4)
6. ordinal collapsing or projection (Section 3.5)
   This is a variety of techniques.
7. minimal standard model for **ZF** (Section 4)
   This is conjectured to be logically determined.

Table 1: Logically determined mathematics

3.2 Infinite AND ($\forall$) and OR ($\exists$) recursively nested

Statements with n alternation[11] between quantifiers over the integers and a recursive relationship on the quantified integers are arithmetical. They are Π_n or Σ_n depending on whether the first quantifier is universal ($\forall$) or existential ($\exists$). These statements are objective.

For example the statement $\forall_{n\in\mathbb{N}}\exists_{m\in\mathbb{N}}\ r(n,m)$ is equivalent to the question of whether a TM easily constructed from r has an infinite number of outputs[3]. This idea can be generalized to a Π_3 statement[12] by asking if a TM has an infinite number of outputs, an infinite subset of which are the Gödel numbers of TMs that have an infinite number of outputs. This idea can be generalized to Π_n statement for any odd finite n.

The nesting of alternating universal and existential quantifiers over a recursive relationship can be iterated up to any recursive ordinals. Hyperarithmetic sets involve transfinite iteration of this idea. This requires a form of diagonalization at limit ordinals. A recursively enumerable sequence of statements of types: $\Pi_1, \Pi_3, \Pi_5, ..., \Pi_n, ...$ is an example of diagonalization that goes past the arithmetical to the hyperarithmetical. Much of the hyperarithmetic hierarchy can be shown to be objective by using Kleene's notations for recursive ordinals[18, 5] to generate the recursively enumerable tree of objective statements the determine the result.

3.3 Well founded trees

To go further requires quantifying over the reals. Define Π^1_n and Σ^1_n statements as those with n alternations of quantifiers over the reals and a recursive relationship between the bound

[11]Statements that have adjacent existential or universal quantifiers can replace the paired quantifiers with a single quantifier by a change to the recursive relationship quantified over.

[12]The sequence obtained by a straightforward iteration of this ides defines Π_n statements with n odd. (Π_n statements include Π_m and Σ_m statements with $m < n$.)

variables. Π_1^1 statements are defined to be objective for the following reasons.

3.3.1 Π_1^1 statements

A Π_1^1 statement is of the form:

$$\forall_{s \in \mathbb{R}} \exists_{t \in \mathbb{R}} \; r(s,t) \tag{1}$$

where $r(s,t)$ is a recursive relationship. For convenience $\mathbb{R}$ is defined to be the reals ≥ 0 and < 1 encoded in base 2. Statement 1 can be transformed to:

$$\forall_{s \in \mathbb{R}} \exists_{k \in \mathbb{N}} \; r'(s, u(k)) \tag{2}$$

where $u(k)$ enumerates the initial finite sequences of all reals each paired with its length. For a particular value of k, r' initially does exactly what r would do with s and a t that starts with the initial segment given by $u(k)$. Eventually either r would terminate with a decision or r would interrogate t past the initial segment length specified by $u(k)$. In the first case r' outputs the result that r would have output and terminates. In the second case r' outputs false and terminates. There exists a t such that $r(s,t)$ holds for a particular s iff there is a k that specifies a long enough initial segment of t that $r'(s, u(k))$ is true. Thus statement 1 is true iff statement 2 is true.

Statement 2 is true iff the finite initial segments of s for which $\exists_{k \in \mathbb{N}} \; r'(s, u(k))$ is true cover all reals. These are the initial segments of s interrogated by r' before r' returns true. All pairs of initial segments of s and t for which $r(s.t)$ is true are recursively enumerable. It is objectively true or false that there is an initial segment of every real in the first element of the these pairs of finite segments. This is not because all reals can be searched. Every set that has the *property* of defining a real must have or not have an initial segment as the first element of every pair in this recursively enumerable collection. There are trivial cases for which this is true. For example the first bit of every binary real is 0 or 1. Thus if the pairs contain two with these first elements, the statement is true. Of course if you know there is a finite covering sequence then you only need to quantify over the integers to get a logically equivalent statement.

One can see the relevance of Π_1^1 sets to physical reality in different ways. For example the set of all notations for recursive ordinals in Kleene's $\mathcal{O}$ is a Π_1^1 complete set[13]. These notations are defined so that given the integer value of a notation One can construct a nondeterministic computer program to enumerate the structure of the ordinal it represents in the form of a well founded tree. Tree structures are essential for understanding physical divergent processes including the biological tree of the evolution of life.

3.3.2 Σ_1^1 statements

A similar argument applies to Σ_1^1 statements. These are of the form:

$$\exists_{s \in \mathbb{R}} \forall_{t \in \mathbb{R}} \; p(s,t) \tag{3}$$

where $p(s,t)$ is a recursive relationship.

[13] A TM with an oracle for a Π_1^1 complete set can decide any Π_1^1 statement.

As before reals are considered to be ≥ 0 and < 1 encoded in base 2. By defining $u(k)$ as enumerating all initial segments of reals paired with the segment length one can define:

$$\exists_{s \in \mathbb{R}} \forall_{k \in \mathbb{N}}\ p'(s, u(k)). \tag{4}$$

For a particular value of k, p' initially does exactly what p would do with s and a t that starts with the initial segment given by $u(k)$. Eventually either p would terminate with a decision or interrogate t past the initial segment length specified by $u(k)$. In the first case p' outputs what p would have output. In the second case p' outputs true. There exists a t such that $p(s, t)$ yields false for a particular s iff there is a k' that yields a sufficiently large initial segment of t that $p'(s, u(k'))$ yields false. Thus Statement 4 is equivalent to Statement 3.

As in the previous section all pairs of finite initial segments for which $p(s, t)$ is true are recursively enumerable. The full statement is true iff there is a collection of pairs with second elements that covers all reals and the corresponding first elements are all initial segments of the same real.

Objective is defined to include Π_1^1 and Σ_1^1 sets. As an example of why the latter should be included consider the question previously mentioned in Section 2: will a species have an infinite chain of descendant species. In a recursive unbounded universe this question is Σ_1^1.

3.4 Ordinal notations with labeled nodes in well founded trees

Kleene's $\mathcal{O}$ can be generalized to provide notation for larger ordinals. The notations in $\mathcal{O}$ represent zero, a successor ordinal or a limit ordinal. A successor notation encodes its predecessor. A limit notation encodes the Gödel number of a TM which outputs an infinite sequence of notations for smaller ordinals in increasing order. This notation for a limit ordinal defines a function from the integers to increasing ordinal notations. The integers implicitly number the TMs outputs. The limit ordinal represented is the union of the ordinals represented by the enumerated notations. Starting from any notation in $\mathcal{O}$, one can recursively construct a well founded tree with limit nodes indexed by the integers. This tree embeds the structure of a recursive ordinal.

Notations in $\mathcal{O}$ can be generalized by defining recursive functions with domains that are not recursively enumerable. $\mathcal{O}$ is the first of these domains[5]. For example a TM which copies its input to its output[14] can be used in a notation for ω_1^{CK}. This function's domain and range are both the notations in $\mathcal{O}$. Thus the union of all ordinals represented in its rage is ω_1^{CK}

Notations based on this idea require two labels to indicate the input and output types for the TM encoded in a notation for a limit ordinal. The first of the two labels gives the allowed input type. The labels themselves are ordinal notations. A first label of 0 signifies the ordinal represented is zero or a successor ordinal and thus had no input parameter. A first label of 1 signifies a limit ordinal defined by a function on the integers. A first label of 2 signifies a parameter that is the notation for a recursive ordinal.

[14]The ordinal calculator[7] mentioned in Section 1.2 does not use the identity function for the notation for ω_1^{CK}. Instead it uses a function that diagonalizes previously defined functions on the integers in the notations for limit ordinals. Its domain and range are both ω_1^{CK}, but its output may be much larger than the corresponding input.

The second label is an inclusive upper limit on the notation type of an output for an input of the type specified by the first label. The second label is necessary in a recursive system like the ordinal calculator. It must be possible to recursively determine which notations are valid parameters for the TM referenced in a limit ordinal notation.

This idea can be generalized to a hierarchy of TM domains with ordinal notation labels. This can be iterated up to any recursive ordinal and beyond. This approach has been formalized in a direct generalization of Kleene's $\mathcal{O}$[5]. A recursive version with the inevitable gaps has been implemented in an interactive ordinal calculator[6, 7].

3.5 Ordinal projection or collapsing

Ordinal collapsing is not a specific technique. It is an idea on which multiple techniques can be based. The essential idea is that one can use the names of larger ordinals to construct notations for smaller ordinals not previously defined. For example, uncountable ordinals can be used to name countable ones and ordinals $> \omega_1^{\mathrm{CK}}$ can be used to name recursive ordinals.

For recursive systems, in which a recursive process can determine which strings define ordinal nation, collapsing techniques can be used to partially fill gaps[15]. A variation of this is used in the ordinal calculator[6].

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4 The minimal standard model for ZF

> [The] 'constructible' sets are defined to be those sets which can be obtained by Russell's ramified hierarchy of types, if extended to include transfinite orders. The extension to transfinite orders has the consequence that the model satisfies the impredicative axioms of set theory, because an axiom of reducibility can be proved for sufficiently high orders—Kurt Gödel[12].

Gödel defined the constructible universe **L** and proved it was a model for **ZF** in which the axiom of choice and the continuum hypothesis were true[14, 15]. Thus, if **ZF** is consistent, **ZF** plus these two axioms is consistent. Sets in **L** are defined in an ordinal indexed sequence $\mathbf{L}_\alpha$. The construction is iterated over all ordinals.

Shepherdson first developed a minimal model for set theory[22]. Cohen used a slightly modified version of Gödel's definition of the constructible universe[8] to define the minimal standard model[16] for **ZF** (see Table 2). Unlike Gödel's constructible universe, it cannot include all ordinals. It is limited to a countable set of objects defined by formulas in the language of **ZF**. The iterative construction has a fixed point at a countable ordinal.

Two properties of the minimal standard model are crucial to showing it is objective. It is countable and every set in the model has a definition derivable from the axioms of **ZF**. It is straightforward (although completely impractical) to program a TM to enumerate all the

[15]For example there is a gap starting with the limit of the recursive ordinals recursively recognized in the system and ending at the ordinal of the recursive ordinals, ω_1^{CK}.

[16]A standard model is well founded and uses the standard set membership definition of $\in$.

1. $T_0 = \omega \cup \{\omega\}$.

2. $C_\alpha = \bigcup_{\beta<\alpha} T_\beta$.

3. $x, y \in C_\alpha \to \{x, y\} \in T_\alpha$.

4. $x \in C_\alpha \to \{y \mid \exists_u\ y \in u \in x\} \in T_\alpha$.

5. $x \in C_\alpha \to \{y \mid y \subseteq x \wedge y \in C_\alpha\} \in T_\alpha$.

6. Define $A_{n,\alpha}(x, y)$ as the nth formula in the language of **ZF** with all constants and bound variables restricted to C_α.
   Define $B_{n,\alpha}(u, v)$ as v is the image of u under $A_{n,\alpha}(x, y)$

   $$B_{n,\alpha}(u, v) \equiv [\forall_{y\in C_\alpha}\ y \in v \equiv \exists_{x\in C_\alpha}[x \in u \wedge A_{n,\alpha}(x, y)]].$$

   Define $\exists!\, y$ as there exists a unique y.

   $$[\forall_{x\in C_\alpha}\exists!_{\,y\in C_\alpha}\ A_{n,\alpha}(x, y)] \to \forall_{u\in C_\alpha}\exists_v\ B_{n,\alpha(u,v)} \wedge v \in T_\alpha.$$

7. Only the sets described in 1 through 6 belong to T_α.

The minimal model is $M = \bigcup_{\alpha \ \mid\ \alpha \text{ is an ordinal}}\ T_\alpha$.

Table 2: Minimal standard model for **ZF**[8, 9]

theorems derivable from the axioms. These theorems will include a definition of every set in the minimal standard model.

In Cohen's construction T_α and C_α are iteratively defined. T_α is constructed from rules 1 and 3 to 7 (see Table 2 or the $\boxed{\text{boxed}}$ rules below) using only sets in C_α which is constructed from $T_{\beta<\alpha}$ in rule 2.

To prove that every set in the minimal standard model is objective requires two conditions. First the operations that define T_α must be shown to go from objective sets to objective e sets. Second the first fixed point in the definition of C_α from $T_{\beta<\alpha}$ (rule 2) must be the limit of objective ordinals. I conjecture that this second condition is true.

Following is the outline of a proof that Cohen's rules that define T_α (all but rule 2) map objective sets to objective sets. Each of these rules is followed by an explanation of why the construction goes from objective sets to objective sets.

$\boxed{\mathbf{1.}\ T_0 = \omega \cup \{\omega\}}$

ω is an objective set.

$\boxed{\mathbf{2.}\ C_\alpha = \bigcup_{\beta<\alpha} T_\beta}$

This is induction up to the ordinal α. Assuming α and $T_{\beta<\alpha}$ is objective, then this is objective.

$\boxed{\mathbf{3.}\ x, y \in C_\alpha \rightarrow \{x, y\} \in T_\alpha}$

A set containing a pair of objective sets is objective.

$\boxed{\mathbf{4.}\ x \in C_\alpha \rightarrow \{y \mid \exists_u\ y \in u \in x\} \in T_\alpha}$

The union of all sets contained in members of an objective set is objective.

$\boxed{\mathbf{5.}\ x \in C_\alpha \rightarrow \{y \mid y \subseteq x \wedge y \in C_\alpha\} \in T_\alpha}$

This rule can generate sets that are not countable from within the model being constructed. They are however countable as seen from outside. Every member of C_α must have a formula that names it. These formulas are recursively enumerable. Thus the union of objective sets contained in C_α that are subsets of x are objective.

6. Define $A_{n,\alpha}(x, y)$ as the nth formula in the language of **ZF** with all constants and bound variables restricted to C_α.
Define $B_{n,\alpha}(u, v)$ as v is the image of u under $A_{n,\alpha}(x, y)$

$$B_{n,\alpha}(u, v) \equiv [\forall_{y \in C_\alpha}\ y \in v \equiv \exists_{x \in C_\alpha}[x \in u \wedge A_{n,\alpha}(x, y)]].$$

Define $\exists!\,y$ as there exists a unique y.

$$[\forall_{x \in C_\alpha} \exists!\,_{y \in C_\alpha}\ A_{n,\alpha}(x, y)] \rightarrow \forall_{u \in C_\alpha} \exists_v\ B_{n,\alpha(u,v)} \wedge v \in T_\alpha.$$

If a statement in the language of **ZF**, $A_{m,\alpha}$, is true for a unique $y \in C_\alpha$ for every $x \in C_\alpha$ then this statement defines a function $f_{m,\alpha}(x)$ to and from C_α. The image of any $z \in C_\alpha$ under $f_{n,\alpha}$ is in T_α.

This image is objective because all sets and operations in $A_{n,\alpha}(x, y)$ are objective. The quantifiers are restricted to members of C_α which are objective and countable. The other operations are either logical or involve set membership or equality. These are objective if their operands are and the operands are restricted to C_α.

$\boxed{\text{7. Only the sets described in 1 through 6 belong to } T_\alpha}$

This constraint is necessary to insure the model is minimal, countable and objective.

5 Uncountable cardinals and incompleteness

If infinite totalities are human conceptual creations, Cantor did not prove that there are more conceptual creations of reals than there are integers. Absolutely uncountable sets cannot be objective relative to the universe assumed here. However, Cantor's uncountability proof combined with the Löwenheim-Skolem theorem imply an incompleteness theorem. They specify a technique for diagonalizing the reals in an objective formal system that defines a stronger objective system. Uncountable hierarchies can be thought of not as specific sets, but a bit like computer program macros or C++ templates. These do not specify computer code by themselves. Macros require parameter values and templates require data types to generate executable code. Cantor diagonalization can map a suitable objective model to a larger objective model.

A form of pluralism is implicit in the many sometimes incompatible ways[17] that mathematics may be expanded. An objective minimal standard model may be extended by either of two incompatible axioms. Without changing the model in which the first axiom holds one may create a larger model in which the second axiom holds. An objective minimal model may be expanded in this way alternating between the two axioms each time creating a more inclusive model with a hierarchy of submodels. Iterating this up to a limit ordinal may lead to a model in which neither axiom holds. When this happens neither axiom may be true or false in an absolute sense although they may both be useful[18]. I suspect that this may be the fate of the continuum hypothesis. It may be neither true nor false in an absolute sense but very useful. Ordinal collapsing to expand notations for recursive ordinals does something similar when they use uncountable ordinal hierarchies to expand countable ones[21].

6 Objective mathematics and philosophy

Objective mathematics incorporates a modern version of the classical idea that infinity is a potential that can never be fully realized. It has the advantage of a deep and rich generalized recursion theory and the ability to do extraordinarily complex and accurate computation. The latter leads to a limited form of Platonism albeit tinged with empiricism. The ideal universe of Plato exists in a limited manner in the physical world. We cannot construct a perfect circle, but π has been computed to more than 10^{12} decimal places with a high degree of certainty that it is correct.

Evolution created biological and cultural legacies from common activities such as counting and measuring. It is what the psychologist, Carl Jung, called archetypal knowledge[17].

[17]For example the axiom of determinacy is not consistent with the axiom of choice.

[18]Asking if either axiom is true may be a bit like asking if the set of all integers is odd or even.

What seems a priori to us is actually the result of an enormous number of biological and cultural experiments that sharpened the human ability to think logically.

This legacy is limited. Mathematicians ask questions that cannot be answered definitively. Mathematics will increasingly need a divergent experimental approach similar to the diversity of biological and cultural evolution. This is reflected in pluralism which is essential in fully exploring the possibilities. However, relative to the universe assumed here, there is an expandable core of created, yet logically determined and thus objective, mathematics. The core can only be partially formulated, but we may be able to ensure that an unbounded process is sufficiently diverse to fully explore the possibilities.

This philosophical approach suggests that time is fundamental and irreducible. The most important feature of our universe is its enormous creativity over time. This creativity may be unbounded. It may always be possible to explore more interesting and practically important mathematics. Something similar may be true of the depth and richness of conscious experience given the close connection that has been observed in limited experiments between the structure of the brain and human consciousness.

Perhaps whatever exists at *any* moment in time is the merest hint of a shadow of what can be and that will always be the case[2].

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