Mathematical incompleteness and objectively guided intrinsic values

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Abstract

As science and technology grow at an accelerating rate they create a similar growth in our power to manipulate the physical world. This growth is *only* possible through the *objective* guidance of mathematics and experiments. This has led to many great advances. However the use of this power is largely determined by human values and ultimately what is believed to have *intrinsic* value. Here no objective guide exists. This puts us in the position of the sorcerer's apprentice with far more power than we know how to control. The lack of an objective guide to values has helped to create the existential threats of human caused global warming and nuclear weapons. It contributes to many other problems with immense and needless human suffering.

This article argues that intrinsic value exists only in conscious experience. It links consciousness with physical structure by making the simplest possible assumptions consistent with what we know. This is the starting point for an objective understanding of intrinsic value.

Mathematical incompleteness implies that *any* consistent mathematical system, can always be expanded to decide more questions about the unbounded evolution of physical systems. This expansion involves deeper and richer structures for abstraction and self reflection. The Totality Axiom proposed here combined with the evidence from biological evolution suggests the same is true of the evolution of deeper and richer conscious experience.

Perhaps the most important implication is the logical necessity for an ever expanding diversity of approaches if one does not want to limit the power of mathematics or the depth and richness of conscious experience. The search for the *one true path* is futile and counterproductive.

Contents

1 Introduction

Science and technology are changing the world at a rate that is difficult to adapt to. They have created existential threats with nuclear weapons and global warming. They are creating extraordinary opportunities for much of the planet accompanied with immense and needless human suffering[1]. Stopping or slowing the advance of technology is not a realistic option.

There are essential partial solutions to current problems, but there is a fundamental issue that must be addressed. Existing value systems are inadequate. Our spiritual and religious traditions developed in and adapted to times of gradual change sometimes interrupted by clashes between cultures.

Science and engineering made substantial and consistently accelerating progress after they recognized that objective criteria (mathematics and experiments) took precedence over 'revealed' truth, ancient wisdom and compelling but unverifiable arguments[2]. Our value systems need similar objective guides to adapt to the world we are creating.

1.1 Consciousness and physical structure

This article argues that conscious experience is the *only* source of *intrinsic* value (Section 2). It then links conscious experience with physical structure through the Totality Axiom by making the simplest possible assumptions consistent with what we know (Section 2.1). The Totality Axiom is clarified and expanded in the rules of consciouses (Section 3). Next is a brief overview of mathematical truth (Section 4). This forms an unbounded hierarchy (Section 5) which can never be fully explored by finite beings in a finite time although there is no part of the hierarchy that cannot be explored eventually if time is unbounded. This leads to a mathematical derivation of the boundary conditions that allow unlimited development of intrinsic value and consciousness (Section 5.3).

Mathematics proves that the evolution of physical structure is in theory unbounded in the level of feedback, abstraction and self reflection that can be embedded in it. The power of today's computers and the greater power of the human brain are limited examples. The unbounded potential for the evolution of physical structure follows from perhaps the most important mathematical result of the 20th century, Gödel's Incompleteness Theorems. The Totality Axiom transfers this result to the evolution of consciousness[2]. This analysis of intrinsic value establishes its own limitations. Unbounded development requires ever expanding diversity.

[1] Examples of needless human suffering facilitated by technology include not only wars, other conflicts and terrorism, but also cancer caused by smoking, junk food that leads to diabetes, drug addiction, economic depressions, etc.

[2] Ray Kurzweil has argued persuasively that technological exponential growth goes back to the beginning of life on earth as long as one regards the techniques of biological evolution as technology[27]. He argues that exponential growth exists because technology accelerates its own growth. It manages to do this indefinitely by paradigm shifts when one focus of progress starts to stagnate. Perhaps the most notable is the shift from biological to cultural evolution as the primary change agent. The shift to objectivity in modern science and engineering did not create exponential technological growth, but it was a necessary paradigm shift to maintain that growth. Prior to this shift, exponential growth was slower and seemed linear if not flat unless one carefully examined long (relative to the human life span) time periods.

1.2 Threats to civilization and humanity

Many commentators (perhaps most notably Bill Joy[23]) think we may be approaching an increasingly dangerous time. Threats include the following.

1. Climate change from global warming caused primarily by fossil fuels.
2. Nuclear weapons.
3. Economic implosion. As technology and productivity advance, the need for laborers (even highly skilled ones) declines. The result is greater wealth to fewer individuals with world output peaking and then declining as the customer base shrinks. Dealing with this requires a new economic model that provides incentives for talent, skill and productivity while insuring the benefits are widely distributed[19].
4. Biotechnology that may
   (a) be used for warfare,
   (b) make it possible for an intelligent, knowledgeable and disturbed individual to create monstrously dangerous organisms or
   (c) lead to the accidental release of highly dangerous organisms from a research facility.
5. Nanotechnology that may lead to minute reproducing structures that do great damage including possibly the complete destruction of the biosphere.
6. Robotics that will reach the level of human intelligence and then exceed it at a rapidly accelerating pace, perhaps making it inevitable that they will come to control a planet on which human life may seem superfluous and wasteful.

2 Intrinsic value and consciousness

> As regards the world in general, both physical and mental, everything that we know of its intrinsic character is derived from the mental side, and almost everything that we know of its causal laws is derived from the physical side. But from the standpoint of philosophy the distinction between physical and mental is superficial and unreal. — Bertrand Russell[29, p. 402]

In ancient philosophical traditions there were fundamental elements with an intrinsic nature such as earth, air, fire and water. Combinations of these substances created other substances and their intrinsic nature. Later some physicists thought that atoms were billiard ball like fundamental particles that gave substance and an intrinsic nature (such as soft or hard) to objects made of them. In 1927, when Bertrand Russell published the above quote, quantum mechanics and relativity were converting fundamental physics to purely mathematical models lacking any intrinsic nature.

The lack of intrinsic nature in science and mathematics is made explicit in set theory from which almost all accepted mathematics can be derived. In set theory there is only

one element not defined in terms of other elements and that is the empty set or nothing at all. The first object built from this is the set containing the empty set. Set theory models structure devoid of substance and physics has become entirely mathematical.

2.1 The Totality Axiom

How can conscious experience with an intrinsic nature connect to physical structure? The key to what has been called the hard problem of consciousness[8] may lie in Bertrand Russell's observation combined with the assumption that everything that exists has an intrinsic nature. It *feels* like something to *be* something. What would it mean to exist without having an intrinsic nature?

These assumptions imply that everything is consciousness in some form. Of course consciousness must come in many simpler forms than human stream of consciousness. It can be as simple as a single point in the visual field.

It is important to distinguish between this universal consciouses and human stream of consciousness. For the latter the term 'aware' and the phrase 'stream of consciousness' will be used. The term 'consciousness' and the phrase 'immediate experience' refer to universal consciousness which includes human stream of consciousness. This distinction is not an absolute division, but more of a continuum. Something may be the intense focus of awareness, barely on its periphery or anywhere in between.

These ideas led me to the Totality Axiom: *Immediate experience in some form is the essence and totality of the existence of physical structure and structure is the only aspect of existence that can be communicated.*

The Totality Axiom is a form of panpsychism or the belief that consciousness is universal in all that exists. In its animistic form, panpsychism is thoroughly discredited by contemporary science. But it exists in more abstract forms as the artificial intelligence researcher and futurist Ray Kurzweil has suggested.

> So we could say that the universe — "all that is" — is indeed personal, is conscious in some way that we cannot fully comprehend. This is no more unreasonable an assumption or belief than believing that another person is conscious. Personally, I do feel this is the case. But this does not require me to go beyond the "mere" "material" world and its transcendent patterns. The world that is, is profound enough[26, p. 215].

The mythologist, Joseph Campbell, had a similar sense of the universality of consciousness.

> It is part of the Cartesian mode to think of consciousness as being something peculiar to the head, that the head is the organ originating consciousness. It isn't. The head is an organ that inflects consciousness in a certain direction or to a certain set of purposes. But there is consciousness here in the body. The whole living world is informed by consciousness.
>
> I have a feeling that consciousness and energy are the same thing somehow. Where you really see life energy there is consciousness. Certainly the vegetable

> world is conscious. And when you live in the woods as I did as a kid, you can see all these different consciousnesses relating to themselves. There is a plant consciousness and there is an animal consciousness, and we share both these things. You eat certain foods, and the bile knows whether there's something to go to work on. The whole process is consciousness. Trying to interpret it in simply mechanistic terms won't work[7, p. 18].

The philosopher, David Chalmers, has proposed a tentative theory of consciousness based on information. In context Chalmers' information is almost a synonym for mathematical structure. Every structure contains information and any finite structure can be fully described using information. After outlining his ideas he observes that information is ubiquitous. He does not shrink from the conclusion that experience must also be ubiquitous.

> If this [experience is ubiquitous] is correct then experience is associated with even very simple systems. This idea is often regarded as outrageous, or even crazy. But I think it deserves a close examination. It is not so *obvious* to me that the idea is misguided, and in some ways it has a certain appeal[9, p. 293].

2.2 Universal consciousness

The awareness we remember and describe is directly connected to memory and language. That is not necessarily the only immediate experience in the spatial boundaries or our body. Many people believe that at least some animals are conscious. Where does consciousness begin in the biological hierarchy? The simplest assumption is that no boundary exists. Having (or more correctly being) immediate experience in some form is what it means to exist. This would be hard to disprove. The contrary assumption must justify the boundary. What are the assumptions that explain it and what is the basis for accepting those assumptions?

Accepting the possibility that consciousness is universal and embodies the intrinsic nature of all existence leads one to ask: what else, if anything, exists? Consciousness contains some of the structure that science describes. Vision is an obvious example. It embodies the physical structure of the pattern of light on our retina. Our visual awareness usually focuses on the objects we recognize, but we can shift it to see patterns of light. Consciousness embodies some physical structure. If it is universal, it could embody all structure. There is no need to assume anything exists but structured consciousness.

The Totality Axiom's assertion that structure is the only aspect of existence that can be communicated may seem obviously wrong. A vibrant description can come alive as a well crafted movie does. Within the universally accepted theory of information developed by Claude Shannon in the 1940's[30], it is only structure or more precisely state that can be communicated. Information is anything that allows one to reduce the number of states that a system may be in. For example assume a signal may be red, yellow, green or not working. If one is told it is red, the possible states have been reduced from four to one. This requires two bits[3] of information with four possible states (00, 01, 10 and 11). This information may evoke the image of a red light, but that is a response to the information and not inherent in it.

[3] A bit is a marker that has only two alternatives such as the digits 0 or 1.

A movie provides visual and audio experience directly by stimulating our senses and indirectly as we connect it to previous experience. The information in many movies is a huge string of bits. DVDs contain nothing but bits. Many theaters obtain their movies as bit streams over a high speed Internet connection.

Showing a movie involves using the information in a bit stream to create the physical stimuli of images and sound. The bit string encodes the color and brightness of each pixel (or small region of an image) every 1/60 of a second and what the sound pressure level is every 1/44,000 of a second [4]. The awareness that results from these sensations cannot be communicated. For example, the totally color blind from birth cannot experience the sensation of color no matter how one attempts to communicate it. They may get a poetic understanding of color and how it may affect other people, but the experience of seeing a colored scene is impossible to communicate.

2.3 The wonder of consciousness

As a small child I marveled at the most mundane of sensations. Green grass, blue sky, the smell of freshly cut weeds, they were astounding. Where did these experiences come from? Why could I have them? Perhaps there is no answer. Even as a child the claim that 'God created it' was not satisfactory. The simplest assumption is that nothing exists, but it is obviously wrong. Everything that could exist does exist is the next simplest assumption, but it seems inconsistent with the creativity of an evolving universe. This suggests the third alternative. Everything that could exist does exist or will come to exist. This can never be proven false although, exactly what it means, is not clear.

The history of evolution combined with contemporary mathematics and science suggests that what could exist is unimaginably beyond what is. Reproducing molecules have evolved to the depth and richness of human consciousness. Is the evolution of consciousness near a limit? Mathematics implies that there is no theoretical bound to the evolution of structure. The Totality Axiom combined with this mathematics opens the possibility that there is no limit to the evolution of consciousness. Current cosmology suggests there are time and size limits to the evolution of structure, but they are very large and highly speculative bounds[5].

The fundamental constraints for unbounded development are ever increasing diversity combined with ever increasing resources for individuals. This requires ever increasing total resources and cannot continue forever on a finite planet. However, science and technology may provide sustainable growth in resources for far longer than one might think possible. By then we may have mastered interstellar travel at least for robotically manned probes containing much of human knowledge and capable of recreating life if a suitable habitat is found or can be created using resources on the probe and the planet being evaluated.

[4] Both video and audio sampling rates vary within an acceptable range based on the capabilities of human sight and hearing.

[5] We know almost nothing about dark matter and dark energy. Many physicists think they make up over 95% of the mass of the universe. Cosmology may be wrong about the fate of the universe. Its previous predictions have changed dramatically with more data and a deeper understanding.

3 Proposed rules of consciousness

Science aims for the the simplest theory that explains as much as possible. The Totality Axiom exists on the borderline between science and philosophy. It aims to be the simplest theory that explains as much as possible, but it depends on subjective reports. Brain scans can, up to a point, give objective validity to subjective reports. For example, there is a limited ability to translate brain scans to an image of what a person is experiencing visually[28].

The Totality Axiom leaves open many questions about the relationship between physical structure and conscious experience. The following proposed rules answer some of these questions with the simplest assumptions consistent with what we know.

1. Just as structure can have multiple levels of organization (molecules, cells, organs, neural networks, etc.) so can conscious experience.

   Our stream of consciousness is at a high structural level. It contains no direct experience of many parts of the neural structure that are essential to facilitate awareness. For example we remember things without any awareness of the neural processes that make this possible[6]. The Totality Axiom implies direct conscious experience exists at all levels of physical structure.

2. Lower level structures affect higher level consciousness only to the degree they affect structure at the higher level.

   This is the simplest possible assumption and will become an empirical question when we are able to replace parts of the brain that directly affect consciousness with non-biological prosthetics that exactly duplicate the function of the part being replaced.

3. Physical structure completely determines conscious experience.

   There are nonstructural aspects of consciousness, for example, the experience of green. One might assume that such experiences are independent to some degree (or completely) of structural constraints. The simplest assumptions is that they are not.

4. Identical structures are identical experiences.

   Your perfect clone, in an exact duplicate of your environment, would have the same experience you do. This follows from rule 3.

5. Isomorphic structures have (are) the same experience at the level of the isomorphism.

   Perfect duplication of the *functioning* of your neural circuits, even with different technology such as electronics, combined with a perfect simulation of your environment, would duplicate your stream of consciousness. This follows from rules 2 and 3.

6. Changes in physical structure are isomorphic to changes in conscious experience.

   We are not aware of most of the changes in our neural network. This does not mean these changes are not conscious, but this consciousness is not directly connected to

[6]We become aware of memory as a process when it fails. For example, the word we are searching for may eludes us.

memory and language. This follows from the assumption that the *essence and totality* of physical structure is conscious experience.

7. The depth and richness of conscious experience associated with a physical structure is limited by the level of abstraction and self reflection embeddable in the structure.

   This is based on mathematics and empirical evidence. The level of ordinal induction (Section 5.2) provably valid within a mathematical system in part determines what questions are decidable and what structures are definable within the system. The ordinal numbers form a hierarchy of abstraction and self reflection. The human mind seems capable of rich and deep experience because, in part, of the high level of abstraction and self reflection that it is capable of.

The foundation of mathematics is essential to understanding the implications of rule 7.

4 Mathematical truth

Mathematics is the only scientific field with absolute truths like 2+2=4. Among these is a proof that mathematics is and *always will be* incomplete. Kurt Gödel, in his incompleteness theorems, proved that any consistent formal system[7] in which one can embed the basic rules of arithmetic must be incomplete in the sense that there are mathematical questions definable but not decidable in the system[17].

All finite questions can in theory be decided[8] within a single mathematical system. However, the rules of arithmetic include statements about *all* integers (n) such as: '$(n+1) > n$.' The consistency of a formal system can be defined as a question about all integers[9] and it is this question about itself that is definable but undecidable in any consistent formal system that includes the basic laws of arithmetic. One cannot get around this limitation by strengthening a system. It is possible to prove the consistency of a weaker system using a stronger one, but one cannot prove the consistency of either system, if it is consistent and includes elementary arithmetic, within itself.

[7] A formal mathematical system is a set of rules for proving mathematical relationships called theorems. The rules include universal laws of logic and axioms specific to this formal system. These rules must be precise enough to define a computer program for an ideal computer, i. e. one with unlimited memory and the ability to run error free forever. Such a computer can be programmed to list all the theorems deducible from the rules. Other definitions have been proposed that allow an infinite number of axioms that cannot be generated by a computer program and/or rules of inference that are not computable. These alternatives are not considered here although Gödel's proof applies to them as well. If the system is inconsistent, every statement in the system and its negation are both provable and thus Gödel's proof does not apply.

[8] Finite questions can be so difficult that, although we know a method guaranteed to correctly decide them them, it is not practical to apply this method to some specific problems with today's or any foreseeable technology. Determining the prime factors of very large integers is one example on which public key cryptography is based.

[9] One can write a computer program that systematically checks every statement in a formal system to see if it contradicts any previously checked statement. The process will find a contradiction in a finite time if one exists. If not the program will just keep running with no result. Thus to say a formal system is consistent is to say the program will not find a contradiction in n steps for all integers n.

4.1 Self evident and creative mathematics

There can be no systematic way to decide all mathematical questions posed in a formal system that includes arithmetic, but there is a systematic way to explore all plausible axioms that might decide these questions. This may seem impossibly difficult, but something like it has happened. Immense diversity of life, spanning over 4 billion years, and immense complexity in the nervous system as it evolved has created the mathematically capable human mind. This combined with thousands of years of cultural evolution has created mathematics.

A bounded finite number of approaches to developing mathematics has what I call a "Gödel limit". Within the limit, progress can be made forever, but the entire sequence of results obtained over an unbounded time can be fully captured in a finite axiom which will never be explored. Only a divergent process, following an ever increasing number of paths, can avoid a Gödel limit[10].

Most mathematicians think elementary arithmetic is self evidently true. There is a level of mathematics that the talented educated mind can understand and usually sees as self evident, albeit with a healthy respect for human fallibility. This ability is an evolutionary and cultural legacy. Many objective mathematical questions are not self evident and many of these are not decidable within accepted mathematics. For example, determining if a computer program will *ever* do something specific like halt (this is called the computer halting problem), can have no general solution.

4.2 Computer halting problem

The computer halting problem is related to Gödel's second incompleteness theorem because determining the consistency of any recursive[11] formal system is equivalent to a specific computer halting problem. Almost anyone who uses a computer regularly has had experience with this problem. The computer is not responsive. Is this temporary or permanent? You can wait and hope or reboot and perhaps lose work. If the computer does not have a hardware problem, one can in theory determine exactly what the program will do at each point in time[12]. However, there is no general way to decide if a program will *ever* do something specific such as prompt for more input.

The proof is straight forward if one accepts without proof that there is a universal computer that can simulate all other computer programs. This follows from Church's Thesis[11, p 356] which is a definition of effectively computable that has become widely accepted. It is not an assumption of Gödel's proof. He proved that arithmetic, as a formal mathematical system, can model itself and then used an argument that is somewhat similar to the one that follows.

[10] If unlimited resources are available and there is no need to select which approach is correct, all possibilities can be explored, each with ever expanding resources needed to deal with continually increasing complexity as longer statements are explored. All alternatives that cannot definitely be excluded must be allowed.

[11] Recursive means computable or capable of being generated by an ideal computer.

[12] Computer operating systems are not as deterministic as the idealized computer. They operate on different levels of code in response to external conditions. This is designed to be transparent to application programs, but it can be an added source of problems.

Any of today's computers qualify as a universal or ideal computer with the caveat that there must be no limit to the memory the computer has access to and it must be error free for as long as needed. Programs for the universal computer can have a unique integer assigned to them as a Gödel number[13]. All of today's computer programs are a sequence of binary bits that define a Gödel number that uniquely identifies the program relative to the specific computer it is intended to run on.

4.2.1 Assume the halting problem is solvable

Assume there exists a computer program with Gödel number p that has a single integer input x and outputs 0 or 1. The output of p for input x is written as $p(x)$. Assume x can be the Gödel number of *any* computer program for a specific universal computer. Finally assume $p(x)$ is 1 if the program numbered x halts and 0 otherwise. Program p must generate one of these two outputs in a finite time. Following is a proof that this assumption leads to a contradiction and thus there cannot exist a computable solution to the halting problem.

4.2.2 The self halting problem

The first step is to to construct $s(r)$ that solves the self halting problem. This problem asks if a program $r(r)$ will halt. The halting problem operates on programs with no parameters. The self halting problem operates on programs with a single parameter that is equal to the programs Gödel number.

First we prove that, if the halting problem is solvable, then the self halting problem is solvable. For this we show how to construct a program with Gödel number r' that halts if and only if $r(r)$ halts. This construction is computable and can be used, with p, to solve the self halting problem. The modification is to construct r' so that it includes r as both a program and an integer. Further we construct r' to act on this embedded number just as r does with a parameter. The halting problem for r' is equivalent to the self halting problem for r and the construction of r' from r is computable so the self halting problem is solvable if the halting problem is.

4.2.3 The contradiction

Assume the Gödel number for the solution to the self halting problem is s, i. e. $s(x)$ outputs 1 if $x(x)$ halts and 0 otherwise. The next step is to construct s' that behaves like s except if s decides that its parameter is a computer program that halts when presented with its own Gödel number than s' loops forever. If s decides that its parameter *never* halts when presented with its own Gödel number as input then s' outputs 0 and halts.

Now consider the question what does $s'(s')$ do? If $s'(s')$ halts it must run forever and if it does not halt then it must halt. This contradiction means the assumption that p solves the halting problem is false.

[13]Gödel's proof depends on the assignment of a unique integer to every statement defined in the formal system (Gödel numbering). This allows the system to be modeled by arithmetical formulas. The formulas map integers to integers. and these integers are codes for statements that can be provable theorems. 'Gödel numbering' is now used to indicate the assignment of unique numeric codes for objects in many contexts.

4.2.4 Comparison with Gödel's proof

Gödel's proof is different from and far more complex than this argument. He proved a related, but more general result, that no consistent formal system in which one could embed basic arithmetic could prove its own consistency.

Part of Gödel's proof was showing that every statement in a formal system could be assigned a Gödel number and the process of proof could be defined arithmetically. Using this he constructed a model of the formal system within itself. He then proved the consistency of a formal system was a problem in arithmetic. From that he was able to prove, with an argument somewhat similar to the above, that there is a contradiction from the assumption that any consistent system that embeds basic arithmetic proves its own consistency.

5 Hierarchy of mathematical truth

Mathematical incompleteness has wide ranging implications. It is a fundamental constraint on biological evolution. Our nervous system evolved, in large part, to control our actions and predict (not necessarily with awareness) their consequences. For example, if you practice jumping several feet and trying to land in a small area at different distances your body will learn how hard to push and at what angle. You do not think about the mathematics, but your body learns what to do. Controlling our actions and predicting their consequences is crucial to survival. It is almost certainly the primary reason we evolved large (relative to body size)[14] and demanding brains. Human brains consume about 20% of the calories we burn at rest.

Of course the practical problems of finite beings are finite and thus in theory solvable. The general method for solving them is to fully simulate the situation and observe what happens in a finite time. This is rarely practical because of the complexity of the problem or unknowns that cannot be modeled. Something more efficient than a full scale simulation is required for almost all practical problems.

Nature has evolved many ad hoc solutions to classes of problems like coordinating neural signals to muscles to control physical movement. These solutions fall in a hierarchy of mathematical truth. Most of nature's solutions are at the low level in this hierarchy that is defined by the basic laws of arithmetic. However those mental processes that enable mathematicians to understand higher level mathematics must be, in some sense, at the level of that mathematics. Since skill at higher level mathematics had no innate survival value until recently, the complex neural structures that can be developed and trained to understand this mathematics must have evolved for other purposes. I suspect that they largely evolved to help us deal with our fellow creatures. This was an evolutionary advantage for complex and subtle thought and intuition that helped our ancestors cooperate and compete. One result is an evolved mind with the capacity to develop a hierarchy of mathematical truth through cultural evolution.

[14] Chimpanzees and Bonobos are the nearest living relatives to humans. Their bodies are about two-thirds the size of humans but their brains are roughly one-third the size of human brains.

5.1 Induction on the integers

Central to the hierarchy of mathematical truth are proofs that all objects with a particular property (such as being an integer) have some other property. Induction on the integers is the rule for this in basic arithmetic.

Induction on the integers says if the following two conditions hold for property P:

1. $P(0)$ is true and
2. for every integer i, $P(i) \rightarrow P(i+1)$

then for every integer j, $P(j)$ is true.

5.2 Induction on the ordinals

Gebhard Gentzen[20, 21] proved that the axioms (or rules) of basic arithmetic are consistent by using a principle that goes beyond basic arithmetic, induction up to the ordinal ϵ_0[15]. The ordinal numbers are the backbone of the hierarchy of mathematical truth. They generalize induction on the integers.

Ordinal numbers are defined as sets. The first ordinal is the empty set. It defines the integer 0. Every other ordinal is the union of all smaller ordinals. Thus ordinals are ranked by the $\in$ operator. $a \in b$ means a is a member of b and thus $a < b$.

The successor of an ordinal o is the union of all sets contained in o with o itself. The standard notation for the successor is $o+1$. The integers are defined to be the finite ordinals in set theory. This includes 0 and the result of repeating the successor operation any finite number of times. For example 1 is the successor of 0 or the set containing the empty set. 2 is the set containing 1 and 0. The union of all integers, ω, is the first infinite ordinal and the first limit ordinal. Ordinals that are not 0 or a successor are limits.

Induction on the ordinals says that if the following three conditions hold for property P:

1. $P(0)$ is true,
2. for every ordinal $\alpha < \beta \quad P(\alpha) \rightarrow P(\alpha+1)$ and
3. for every limit ordinal[16] $\alpha < \beta \quad \{\forall_{\lambda<\alpha} P(\lambda)\} \rightarrow P(\alpha)$

then $P(\alpha)$ is true for all $\alpha < \beta$.

Some set theorists think infinite sets exist in a Platonic universe of mathematical truth. Historically the infinite was seen as a potential that can never be realized, but the hierarchy of types of infinity (or cardinal numbers) developed by Cantor and the success and power of Zermelo Frankel set theory, that rigorously formulates this hierarchy, has led many logicians to adopt either a Platonic philosophy or a mathematical version of the shut up and calculate philosophy of some physicists unhappy with the foundations of quantum mechanics. The

[15] In basic arithmetic one can define each function in the sequence: $a_1(x) = x$, $a_2(x) = x^x$, $a_3(x) = x^{x^x}$, $a_4(x) = x^{x^{x^x}}$,..., but not the function $b(n) = a_n(n)$. This function can be defined using the ordinal ϵ_0.

[16] Note in the following $\{\forall_{\lambda<\alpha} P(\lambda)\}$ says for every ordinal λ that is less than α $P(\lambda)$ is true.

meaning and philosophical significance of infinite sets or structures remains controversial in mathematics and philosophy. For more about this see[15, 14, 3].

There is a core of mathematics that extends beyond arithmetic that most mathematicians accept. This core includes some or all of the recursive ordinals[17]. They were first defined independently by Kleene[25] and Church[12, 13]. Prior to this a substantial number of these ordinals were defined in detail by Veblen[31]. For more about this see [4, 5]. Loosely speaking the recursive ordinals are those whose structure can be modeled by a computer program.

5.3 Consciousness and the hierarchy of mathematical truth

In the hierarchy of mathematical truth, the strength of a formal system is measured in part by the level of ordinal induction that is valid in it[18]. This is a structural limit on the power of abstraction and self reflection usable in a mathematical system. It can mirror similar limits in the structure and capabilities of neural processes. Such structures support the ability to control our actions, predict the consequences of possible actions and influence others. Biological evolution suggests expansion of these capabilities involves deeper and richer conscious experience.

It is a big leap to go from hierarchies in formal mathematics to hierarchies in conscious experience. However, mathematics is the study of all *possible* structure. Thus, if one considers the Totality Axiom and rules of consciousness (Section 3) to be plausible, then physical structure may be conscious experience whose intrinsic nature is determined by the abstract mathematical structure mirrored in the physical structure of a conscious being.

5.4 Practical implications

The practical implications of mathematical incompleteness may seem relevant only in the future. Science and engineering either have all the mathematics they need or can develop it in the boundaries of existing formal systems[18]. Of course we are a long way from an in depth understanding of the human brain and nervous system. This scientific problem may be partially beyond existing accepted mathematics. Although there is no immediate scientific need for diversity in exploring the foundations of mathematics, the idea of a Platonic multiverse has been proposed as a way of introducing diversity into foundations[22, 6].

The need for diversity in exploring practical possibilities is true now because of the many unknowns in important problems. No matter how good we become at minimizing these unknowns, expanding diversity will always be important. The belief that a specific set of laws or principles determines the *only* legitimate or moral possibility often causes or contributes to conflict and needless suffering. It is ironic that the *absolute* logic of mathematics proves the necessity of ever expanding diversity to fully explore mathematical truth, physical possibilities and the evolution of consciousness. I suspect, in limited ways, the need for balanced

[17]The ordinal of the reclusive ordinals is not recursive. Thus mathematicians that accept all the recursive ordinals generally accept larger ordinals as well.

[18]Formal systems with only recursive ordinals can have an absolute upper bound on the level of ordinal induction they define. However no recursive formal system can define induction on *all* recursive ordinals. Thus a recursive system that defines ordinal induction on the ordinal of the recursive ordinals or larger ordinals defines an ordinal hierarchy with gaps.

diversity and focus applies to creative evolution wherever it occurs. This includes:

- the life experience of an individual, a family or other small groups,
- the growth of a business where focus on a single approach has often worked wonders for a while, but ultimately led to business failure,
- the political and cultural evolution of cities, states and nations,
- the political and cultural evolution of the planet and
- biological evolution.

The creative divergence needed to fully explore mathematical truth, the creativity of biological evolution, the evolution of the physical universe and the creativity that plays a central role in so many human institutions, all suggest that creativity over time is a central feature of the universe. This raises questions about Platonic, absolute, complete and static mathematical truth.

5.5 Creativity and Platonic mathematical truth

Is mathematics created or discovered? One might think that objective mathematics can only be discovered. This is true of finite questions about structures small enough to be modeled physically. There is a changing middle ground of large finite structures that we can reason about, but not model in full detail. All mathematics involving only finite structures, whether or not they can be simulated with existing technology, is considered to be objective in this paper. Finally there are infinite structures.

Many properties of an integer (such as being even) are finite and objective for each individual integer. Yet an infinite collection of all integers that satisfy the property is a human conceptual creation. Thus mathematics of the infinite can be both objective and creative.

Which mathematical statements are objectively true or false relative to the universe we appear to inhabit? They may include statements that are determined by events all of which could occur in an always finite but possibly unbounded universe. Thus, as a philosophical principle, I propose that an objective statement is either finite or logically determined by a recursively enumerable sequence of objective statements[19][3]. "Logically determined" is an expandable philosophical principle that has a partial mathematical definition. For examples the logical 'and' and the logical 'or' of a recursively enumerable sequence of objective statements is logically determined. The same is not true for 'exclusive or.'

Creativity by definition involves change over time. For static mathematics time is another dimension. For consciousness and creativity, time is fundamental.

[19]By this definition arithmetic and hyperarithmetic statements are objective. They can be obtained by iterating the property that a TM has an infinite number of outputs, it has an infinite number of outputs an infinite subset of which are the Gödel numbers of TMs with an infinite number of outputs, etc. The arithmetic statements are obtained by finite iterations of this. The hyperarithmetic statements are obtained by iterating it up to any recursive ordinal. The property can be extended to the set of all hyperarithmetic sets and some more complex sets because the set of all notations (as defined by Kleene) for recursive ordinals is Π^1_1 complete. It cannot extend to analytic statements with more than one alternation between quantifiers.

5.6 Time and mathematical creativity

> If we compare Hilbert's axiom system to Euclid's,..., we notice that Euclid speaks of figures to be constructed whereas, for Hilbert, system of points, straight lines, and planes exist from the outset. — Paul Bernays[1]

Mathematics started as *processes* for counting and measuring. These were abstracted, expanded and formalized by Euclid and others over 2,000 years ago. Eventually the objects of mathematics became pure abstractions devoid of an intrinsic nature. In mathematics time is like an additional spatial dimension. Yet there is an enormous difference in our experience of time and space. Temporal evolution is central to our conscious experience and to mathematics as a creative process.

In the Platonic philosophy of mathematics there is an ideal reality that includes all mathematics in an absolute, timeless and perfect form. This philosophy developed in part because of the huge historical discrepancy between the apparently unbounded accuracy of mathematical reasoning compared with the very limited accuracy of physical measurements and observations. Today that difference has almost reversed. Computers costing less than $1,000 can do physical calculations billions of times a second for extended periods with a very low probability of a single mistake. This is a fete that no mathematician or feasible collection of mathematicians can duplicate. Aspects of mathematics approaching the Platonic ideal exist in today's technology.

The foundations of mathematics rests firmly on a legacy of seemingly a priori truth which includes at least basic arithmetic. We will never fully develop what follows logically from that legacy, but we can extend aspects of mathematics only by going beyond the legacy to create new provisional mathematical truth. Mathematics is a part of the creative evolution of consciousness and will take on added significance as consciousness increasingly controls its own evolution. Fundamental to that creative evolution is the balance between diversity and concentration of resources.

6 Balancing diversity and complexity

Balancing diversity and complexity is a fundamental requirement for climbing the mathematical hierarchy of truth. That balance seems to a be fundamental to a wide range of creative processes.

6.1 Reproductive and life strategies

Carl Jung, in defining the modern usage of the psychological terms, intravert and extravert, observed that it applies to these psychological dispositions and to the fundamental strategies for reproductive success.

> There are in nature two fundamentally different modes of adaptation which ensure the continued existence of the living organism. The one consists of a high rate of fertility, with low powers of defense and short duration of life for the single individual; the other consists in equipping the individual with numerous

> means of self-preservation plus a low fertility rate. This biological difference, it seems to me, is not merely analogous to, but the actual foundation of, our two psychological modes of adaptation [intraversion and extraversion][24, ¶559].

Jung thought that the potential for both types of behavior was innate in most of us. As we grow we become more comfortable outside of our dominant type and we are more adept at choosing what will be most effective in a given situation.

6.2 Facilitating cultural evolution

Jared Diamond in *Guns, Germs and Steel[16]* observed a similar creative dialectic between diversity and concentration of resources in cultural evolution. He investigated why certain cultures came to dominate the planet while others remained relatively stagnant. A culture dominated by a single ruling elite, like China, inevitably failed to pursue possibilities essential to future development. In contrast, a region, like Africa, with so many small communities, could never marshal the resources needed for certain kinds of progress. Europe had a more productive combination of diversity and concentration of resources with relatively few but large competing nation states.

Europe, with so many of its inhabitants impoverished or enslaved serfs and with such brutality in its colonial conquests, was nothing to admire. It was just the most successful of a bad lot, in a world in which civilizations struggled to adjust to slowly accelerating cultural and technological development.

7 Possibilities

The exponential growth of technology comes with threats and with the potential for an unbounded growth in the depth and richness of conscious awareness.

We are developing technology that directly connects to the human nervous system and consciousness. These techniques give limited hearing to the deaf[10] and limited vision to the blind. This approach can eventually allow direct neural connections to the Internet. We may be able to do an Internet search by an act of will and become aware of the results.

Duplicating, even in limited ways, the sensors and neural connections for sight and sound are difficult but well understood engineering problems. We will be able to go much farther as we get a better understanding of the structure and functioning of the human brain. Two recent major projects focus on this understanding. Their primary aim is to create the tools and knowledge that facilitate the prevention, cure and mitigation of horrible brain diseases like Alzheimer's. However their work can lead to enhancing human mental functioning. The goal of the U. S. brain initiative is to develop the tools to support understanding of the wiring and dynamics of the human brain. The European Human Brain Project aims to simulate much of the functioning of the human brain using yet to be built supercomputers[32].

As we gain a deep understanding of the structure and functioning of the brain, we may be able to design neural computer interfaces that obtain information through the Internet with no intention we are aware of. Yet the results may be used in the analysis of a problem in our stream of consciousness. If the source of this information becomes an issue, we will be aware of where it came from and know how to repeat the query.

We may partially control our evolution by seamlessly integrating neural and electronic circuits. That may give us extraordinary powers of sensation, thought and action. We are also on the verge of directly influencing human biological evolution[20]. In the not too distant future we will use genetic engineering to cure horrible diseases[21]. Once it becomes safe and effective, there will be a desire in many to use it to 'improve' future generations. We will need to decide what changes qualify as an improvement and we will need to insure sufficient diversity in our descendants.

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[20] Of course we have always influenced human biological evolution through mate selection and other behaviors.

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