Incompleteness versus a Platonic multiverse

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Abstract

The Platonic multiverse view is that there are multiple and incompatible concepts of a set with corresponding Platonic universes. For example the continuum hypothesis may be true in some of these universes and false in others. This philosophical view leads to an approach for exploring mathematics that is similar to an approach that stems from the more conservative view that infinity is a potential that can never br fully realized.

My version of this view sees infinite collections as human conceptual creations that can have a definite meaning even if they cannot exist physically. The integers and recursively enumerable sets are examples. In this view infinite sets are definite things only if they are logically determined by events that could happen in an always finite but potentially infinite universe with recursive laws of physics. This includes much of generalized recursion theory, but can never include absolutely uncountable sets. "Logically determined" is a philosophical principal that can be partially defined rigorously, but will always be expandable. In this view uncountable sets can be definite things only relative to specific countable (as seen from the outside) models.

Just as Gödel proved that any formal system embedding basic arithmetic must be incomplete in provability, Cantor's uncountability proof plus the Löwenheim-Skolem theorem prove that any sufficiently strong formal first order system must be incomplete in definability. One can always define more reals. In this philosophical view uncountable sets are guides to how mathematics can be expanded. Thus at different stages or paths of development one might assume different and conflicting axioms about uncountable sets.

Contents

1 Introduction

It is an old and widely held view that the integers and their properties are the only mathematical constructs that correspond to objective reality[4, 5]. However, the simplicity, elegance and power of Zermelo Frankel set theory (ZF) are too much to give up for many mathematicians concerned with the foundations of mathematics. The result is a variety of philosophical interpretations ranging from the strictly formalist[5] to the recent proposal for a Platonic multiverse.

The Platonic multiverse view is "that there are many distinct concepts of set, each instantiated in a corresponding set-theoretic universe"[7]. For example the continuum hypothesis might be true in some of the universes and false in others. This philosophical view leads to a divergent approach to developing mathematics that is in some ways similar to the approach suggested by the proven limitations of formal mathematics in our universe at least as we understand it today.

The universe we inhabit appears to be finite and may be unbounded or *potentially* infinite. Those assumptions provide a framework for deciding which mathematics is *objectively* definable at least relative to our universe. Infinite sets are definite things only if they are logically determined by events that could happen in an always finite but potentially infinite universe with recursive laws of physics. This includes recursively enumerable sets and much of generalized recursion theory. "Logically determined" is a philosophical principal that can be partially defined rigorously[1], but will always be expandable.

For example being a notation for a recursive ordinal in Kleene's $\mathcal{O}$[10] (see Section 4) is an objective property of the integers given a specific recursive Gödel numbering of Turing Machines (TMs). However, quantification over the reals[1] is required to define $\mathcal{O}$ which is Π_1^1 complete. In this view relatively uncountable sets can be definite things in a specific countable (as seen from the outside) model. In contrast no absolutely uncountable set can be a definite thing. Cantor's proof that the reals are uncountable combined with the Löwenheim-Skolem[2] theorem prove any sufficiently strong first order system must be incomplete in the reals it defines.

The philosophy of the potentially infinite leads to a similar approach to exploring mathematics as the Platonic multiverse. In either case, one may use different and incompatible techniques for expansion. In the case of the potentially infinite one can alternate between possibilities. The continuum hypothesis may or may not be true in a specific countable model. Whichever is the case, one may be able to further expand the model so the opposite is true.

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[1] Quantification over the reals can have a definite meaning even if the set of *all* reals does not. Consider the assertion $\forall_{r\in\mathbb{R}} S(r)$ where S is a recursive relation on the real r. Because one must be able to compute the result of a recursive relationship in a finite time, a recursive process that computes $S(r)$ can only look at a finite initial segment of the real r before determining if $S(r)$ is true. Thus the set of all initial segments of reals for which S must hold is recursively enumerable. These events collectively determine if S holds for all reals.

[2] The Löwenheim-Skolem theorem implies that any formal first order system that has a model must have a countable model.

2 Objective mathematics

It is convenient to talk of infinite sets as if they were definite things. Many of them are either absolutely or relative to a particular Gödel numbering of TMs. For example the members of a recursively enumerable set are precisely defined. The Gödel numbers of TMs that *do not* halt are definite because they are the complement of a set that is recursively enumerable. Kleene's $\mathcal{O}$ is a precisely defined Π_1^1 complete set. In contrast, because of the Löwenheim-Skolem theorem, we know no uncountable set can be specified as a definite thing in the universe assumed here. We have an intuitive sense of what an arbitrary infinite sequence of digits to the right of the decimal point is, but we can not give this a rigorous definition. In contrast the set of *all* the *finite* initial segments of every real number are recursively enumerable and thus so is what a TM does at every time step for every real number as an oracle.

Central to this approach is recognizing the objectivity of properties of divergent recursive processes. These are definable using a nondeterministic Turing machine that divides its time between an always finite, but possibly ever expanding, number of tasks. (These tasks must be recursively enumerable.) For example consider a finite but potentially infinite universe with recursive laws of physics and an evolutionary process in which species and descendant species are precisely defined. The question will any species have an infinite chain of descendant species for all possible universes and initial conditions can, when suitably formulated, define a Π_1^1 complete set. Everything that ever happens in such a universe can in theory be computed by a single nondeterministic TM that runs error free forever.

Mathematics has roots in counting and measuring. These abstractions apply to all of physical reality. The idea of treating them as abstractions that define techniques that can be applied in many situations is very powerful. Like anything, abstraction can be carried too far. When mathematics moves far beyond abstractions that are in theory *directly* applicable to the physical world as assumed here it is leaving the domain in which objectivity is clearly defined and perhaps definable.

Cantor's proof that the reals are not countable may be the most prominent example of this. Infinite sets are a human conceptual creation that can be quite useful, but cannot always be taken literally. Cantor argued that just as finite sets can be ranked in size so can infinite ones. This claim assumes infinite sets exist in some objective sense. If they do its not in our universe, at least as we understand it today.

Whatever else one might think of Cantor's proof, it is, when combined with the Löwenheim-Skolem theorem, an incompleteness result. Gödel proved that any sufficiently strong formal system can always be expanded in provability. Cantor and Löwenheim-Skolem proved that any sufficiently powerful first order system can always be expanded in definability by adding reals.

Another example of taking the literal existence of infinite totalities beyond what is meaningful in our universe involves using reverse mathematics[3]. The title of a paper, "Finite functions and the necessary use of large cardinals"[6] suggests that there is an indispensable use of large cardinal axioms. This follows because there are theorems about the integers that

[3]Reverse mathematics attempts to find the minimum or simplest formal system in which a theorem is provable.

can only be proven today using large cardinal axioms. The problem is in suggesting that the combinatorial power of proof in a formal system supports an existential interpretation of objects definable in that system. The hierarchies of provability and definability are somewhat independent. The consistency of a formal system is Π_1. This is low in the hierarchy of definability yet there can be no general solution for this class of problems. Large cardinal axioms can decide some of these problems and other properties of the integers that are not *currently* decidable in system with weaker definability. That, by itself, cannot establish the objectivity of the sets defined in these systems. The reason these problems are not decidable by axioms about recursive processes is their combinatorial complexity. Large cardinal axioms may be useful as a first approach to these problems, but an ultimate goal should be an explicit understanding the recursive processes that these formal systems implicitly define.

3 Gödel and a priori knowledge

A classic argument in the philosophy of mathematics concerns a priori knowledge independent of experience. The mind, like the body, is in large part, the product of several billion years of evolution. Thus we have developed innate modes of thinking that have great survival value. Carl Jung gave an analysis of two rational modes that he called thinking and feeling and two pattern recognition modes he called intuition and sensation in *Psychological Types*[9]. Inherited modes of thought and perception combined with cultural evolution has created a sense that mathematical truth is absolute. The educated mind may feel it can directly perceive this truth. This is giving an absolute interpretation to a legacy of biological and cultural evolution. There is deep knowledge independent of an individual's life experience, including what Jung called archetypes[4], but there is no need to assume knowledge independent of *all experience*.

What mathematicians believe about mathematical truth is important to the degree it influences what problems they work on and how they approach those problems. Mathematics as the discovery of eternal truths leads in a different direction than mathematics that creates new conditional truths. As mathematics develops we must transition form the former to the later. We need a different more creative approach to take us beyond the mathematics we can understand directly through our evolutionary legacies.

Gödel proved there were limitations on what mathematical questions can be decided in any consistent and sufficiently powerful formal system. However, there is no theoretical limit on the mathematics we can explore. Mathematicians continue to explore the implications of ZF in the absence of any convincing proof that it is consistent. Even if consistent it is not powerful enough to decide many arithmetical questions starting with its own consistency. Thus there is room for exploring more powerful theories which continues partially in the form of large cardinal axioms. This might be thought of as extending mathematics by building from the top.

It is helpful to build from the bottom in parallel with building from the top. The former

[4]Carl Jung defined archetypes as patterns of thought and behavior that are so common and valuable that they have been incorporated in the human gene pool. Unlike the general modes of thought and perception, Archetypes arise to deal with specific common situations. For example Jung felt that the family and religion are areas where archetypes are important[8].

can use computers to partially construct and manipulate the mathematical objects it defines. This can be an essential aid to intuition and insight in exploring combinatorially complex problems. The field of generalized recursion theory builds from the bottom, but it is often developed using TMs with oracles that cannot be directly simulated on real computers. Kleene's $\mathcal{O}$ and its expansion is an approach to generalized recursion theory that can directly take advantage of the combinatorial power of computers. $\mathcal{O}$ can be expanded by considering TMs that operate on nonrecursively enumerable domains starting with members of $\mathcal{O}$.

4 Generalizing Kleene's $\mathcal{O}$

Kleene's $\mathcal{O}$ is a set of integers that encode effective notations for every recursive ordinal[10]. From one of these notations, notations for all smaller ordinals are recursively enumerable and those notations can be recursively ranked. However, there is no general way to decide which integers are ordinal notation nor to rank an arbitrary pair of notations in $\mathcal{O}$.

In the following italicized lower case letters (n) represent integers (with the exception of 'o'). Lower case Greek letters (α) represent ordinals. $\alpha = n_o$ indicates that the α is the ordinal represented by integer n. The partial ordering of integers, '$<_o$', has the property that $(\forall m, n \in \mathcal{O})\ ((n <_o m) \rightarrow (n_o < m_o))$. The reverse does not hold because not all ordinal notations are ranked relative to each other.

Assume a Gödel numbering of the partial recursive function on the integers. If y is the index of a function under this Gödel numbering, y_n is its nth output. Then Kleene's $\mathcal{O}$ is defined as follows.

1. ordinal $0 = 1_o$.

   The ordinal 0 is represented by the integer 1.

2. $(n \in \mathcal{O}) \rightarrow (2^n \in \mathcal{O} \wedge ((2^n)_o = n_o + \text{ordinal } 1) \wedge n <_o 2^n)$.

   If n is a notation in $\mathcal{O}$ then 2^n is a notation in $\mathcal{O}$ for the ordinal $n_o + \text{ordinal } 1$ and $n <_o 2^n$.

3. If y is total and $(\forall n)\ (y_n \in \mathcal{O} \wedge y_n <_o y_{n+1})$ then the following hold.

   (a) $3 \cdot 5^y \in \mathcal{O}$.

   (b) $(\bigcup\{n : n \in \omega\}\ (y_n)_o) = (3 \cdot 5^y)_o$.

   (c) $(\forall n)\ (y_n <_o 3 \cdot 5^y)$.

   If y is a total function on the integers whose range is an increasing ($<_o$) sequence of notations in $\mathcal{O}$, then $3 \cdot 5^y \in \mathcal{O}$ and represents the limit ordinal that is the union of ordinals represented in the range of y.

$\mathcal{O}$ can be generalized using TMs that operate on domains that are not recursively enumerable starting with Kleene's $\mathcal{O}$. In the expanded notation a limit ordinal is represented by a Gödel number implemented as a symbolic expression related to a mathematical notation for the represented ordinal. This expression for a limit ordinal includes the equivalent of the

Gödel number of a TM combined with a pair of labels that give the domain (denoted by the first label) and an upper bound on its range (denoted by the second label) of that TM. The TM, t, from the extended ordinal notation is a recursive function on notations in its domain to notations that are $<_o$ than the bound on its range. The labels allow a recursive process to decide which notations in the extended notation systems are valid parameters of t. The ordinal represented by the notation that includes t is the union of ordinals represented by notations $t(n)$ for all notations n in the domain of t.

The first domain beyond the integers is Kleene's $\mathcal{O}$. With this domain we can use the TM that defines the identity function as a notation for the ordinal of $\mathcal{O}$. This approach is developed in [2]. It has been implemented in the form of a an ordinal calculator[3].

Mathematically there is little difference between a TM with an infinite oracle and a TM that operates on a domain that is not recursively enumerable. However there is a difference in focus. TMs can be programmed to take advantage of the defining properties of the domain. This allows one to build partial simulations of this domain and its operators. One does not need the full domain or even an initial segment of it to do computer simulations. Only examples of the domain are required and those examples may be recursively enumerable and thus part of a computerized system that one can experiment with.

5 Conclusions

> I believe that the vast majority of statements about the integers are totally and permanently beyond proof in any reasonable system. Here I am using proof in the sense that mathematicians use that word. ...
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> In this pessimistic spirit, I may conclude by asking if we are witnessing the end of the era of pure proof, begun so gloriously by the Greeks. I hope that mathematics lives for a very long time, and that we do not reach that dead end for many generations to come.—Paul J. Cohen[5].

I do not think we are facing a dead end but we are no longer limited to "discovering" truths that, with careful and detailed study, seem self evident (at least to the experts in a specific field) because they are implicit in our evolutionary legacies. Ever since Gödel's results, mathematics has been faced by the conflict between mathematical incompleteness and the strong sense that an educated mind can use a priori mathematical knowledge to decide important mathematical questions. In some areas we have exhausted the a priori knowledge legacies of their most elegant and significant results. Further substantial progress requires that we invent possible truths to explore.

We can use the Platonic multiverse as a rational for doing so, but the only reason we need is the irrefutable incompleteness of mathematics. That incompleteness implies that any single path approach to developing mathematics, at least in the universe assumed here, must run up against what I call a Gödel limit. Progress can be without limit but everything that will ever be discovered will be decidable in a single finite axiom system which is never explored.

Uncountable sets are not definite things. Assuming they are could be a bar to progress as the multiverse view suggests while allowing them to be different definite things in different universes. An alternative is to see axioms that define uncountable sets as guides for expanding the combinatorial power of mathematics. These axioms are not necessarily true or false absolutely but only relative to specific countable models.

Our understanding of combinatorics can be enhanced by constructing the proof theoretic[5] recursive ordinals of ZF and ZF plus various large cardinal axioms. The same process should facilitate expanding hierarchies of larger ($> \omega_1^{\mathrm{CK}}$)[6] countable ordinals. At some point it becomes impossible to directly explore the combinatorial complexity involved without the aid of computers. In contrast there is no *direct* way that computers can help us to explore the mathematics of *absolutely* uncountable sets.

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[5] The proof theoretic ordinal of a formal system is the smallest recursive ordinal that one cannot prove is well founded in the system. For systems with enough expressive power, all smaller ordinals have a notation n in Kleene's $\mathcal{O}$ such that $n \in \mathcal{O}$ is provable in the system. The proof theoretic ordinal is a measure of the combinatorial strength of a system. A system with a larger proof theoretic ordinal can often prove the consistency of a weaker one.

[6] ω_1^{CK} is the ordinal of the recursive ordinals.