

What is Mathematics About?¹

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Abstract

As the Platonic philosophy of mathematics is increasingly being questioned, computer technology is able to approach Platonic perfection in limited domains. This paper argues for a mathematical philosophy that is both objective and creative. It is objective in that it limits the domain of mathematics to questions that are logically determined by a recursively enumerable sequence of events. This includes the arithmetical and hyperarithmetical hierarchies but excludes questions like the Continuum Hypothesis. This philosophy is creative in recognizing that Gödel's Incompleteness Theorem implies one can only fully explore this mathematics by considering an ever increasing number of incompatible possibilities without deciding which is correct. This is how biological evolution created the mathematically capable human mind.

Introduction

Mathematics began with counting and measuring as useful procedures for dealing with physical reality. Counting and measuring are abstract in that the same approach applies to different situations. As these techniques were developed and refined, problems arose in connecting highly refined abstractions to physical reality. The circles that exist physically were never the same as the ideal geometric circle. The length of the diagonal of an ideal square could not be expressed in the standard way that fractional numerical values were defined as the ratio of two integers. Mathematical thought seemed to be creating an abstract reality that could never be realized physically.

Plato had a solution to this problem. He thought all of physical reality was a dim reflection of some ideal perfect reality. Mathematics was about this ideal reality that could be approached through the mind. The difficulties with connecting mathematical abstractions to physical reality often involved the infinite. It takes a continuous plane with an infinite number of points to construct the ideal circle or diagonal of an ideal square. Plato's ideal reality seemed to require that the infinite exists. The idea that infinite mathematical abstractions are an objective Platonic reality became the dominant philosophy of mathematics after Cantor seemed to discover a complex hierarchy of infinite sets.

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This hierarchy has its origins in Cantor's proof that there are 'more' reals than integers. Set A is larger than set B if one cannot define a map or function that gives a *unique* member of B for *every* member of A . This is fine for finite mathematics where one can physically construct the map by pairing off members of A and B . It becomes problematic for infinite sets. If A is larger than B it is said to have larger cardinality than B . The smallest infinite set has the cardinality of the integers. Such sets are said to be countable.

Problems with Infinite Sets

The definition of cardinality creates problems because it depends on what infinite maps are defined in a mathematical system. Formal mathematical systems are, in effect, computer programs for enumerating theorems² and thus can only define a countable number of objects. Because all possible maps from integers to reals are not countable, no formal system will contain all of these maps. Thus we have the paradoxical Löwenheim Skolem Theorem. This says that every formal system that has a model must have a countable model. Thus no matter how large the cardinals one can define in a formal system, there is some model of the formal system in which all these cardinals can be mapped onto the integers. However this cannot be done within the system itself. But when one looks at the system from outside one can easily prove this is true because a formal system is a computer program for enumerating theorems. Every proof that some set exists comes at a unique finite time and thus the collection of everything that provably exists is countable.

A major question about the hierarchy of cardinal numbers is whether the reals are the smallest cardinal larger than the integers. The conjecture that this is true is called the Continuum Hypothesis. It has been proved that both the Continuum Hypothesis and its negation are consistent with the standard axioms of set theory. Thus the question can only be settled by adding new axioms and there is nothing remotely close to agreement about how to construct such axioms. On the contrary there is increasing doubt as to whether the Continuum Hypothesis is true or false in any objective sense.

Solomon Feferman, the editor of Gödel's collected works, observed:

I am convinced that the Continuum Hypothesis is an inherently vague problem that no new axiom will settle in a convincingly definite way. Moreover, I think the Platonic philosophy of mathematics that is currently claimed to justify set theory and mathematics more generally is thoroughly unsatisfactory and that some

² It is tedious but straightforward to go from the axioms of a formal system such as set theory and the laws of logic to a computer program that would enumerate every theorem provable from those axioms and laws of logic. This however is not a practical way to generate new mathematics because most theorems would be trivial.

other philosophy grounded in inter-subjective human conceptions will have to be sought to explain the apparent objectivity of mathematics[4].

Alternative Philosophies

The Platonic philosophy of mathematical truth is dominant but not universal. Constructivism demands that all proofs be constructive. It disallows proof by contradiction[1]. The constructivist treats only those mathematical objects that he knows how to construct as having an objective mathematical existence. Social constructivism has recently been applied to mathematics[3]. This approach sees mathematics as a fallible social construction that changes over time. That is an accurate appraisal of the history of mathematics.

The dominant Platonic philosophy and the extreme form of social constructivism are at opposite ends of a spectrum. In Platonic philosophy there is only absolute truth that must be discovered. In extreme social constructivism all truth is relative to some cultural group that creates and recognizes ‘truth’ through a cultural process.

Constructivism sits between these extremes. It accepts constructive proofs as being absolute but only allows truth values to be assigned to propositions for which there is a constructive proof. It rejects the idea that all valid mathematical questions must be objectively true or false.

Rules in some form are a common element in all these approaches. Mathematics based on a Platonic philosophy depends on formal systems which are precise rules or algorithms (in effect computer programs) for enumerating provable theorems from a set of assumed axioms. Constructivists use similar formal systems with the elimination of proof by contradiction. Social constructivism emphasizes that real proofs are never carried out in complete formal detail, that there are many errors in published work and that there is no agreement about the fundamental axioms of mathematics. This suggests that a social process is the primary element in determining accepted mathematical truth at a given period of time. Nonetheless social constructivism depends on the “rules of the game” as providing a foundation for their philosophy. If there were not rules that could be enforced with some, albeit imperfect, consistency in a social milieu there could be no theory of social constructivism.

Creativity Versus Objectivity

Platonic philosophy ignores the marvelous creativity of our universe. Reproducing molecules have evolved to the depth and richness of human consciousness and created the mathematically capable human mind. One can only gasp in dumbfounded wonder at the miracle of it all. Social constructivism minimizes the connection between objective physical reality and mathemat-

ics. It sees mathematical creativity is somewhat or mostly arbitrary like many cultural practices seem to be. Is it possible to square this circle with a philosophy of mathematics that integrates aspects of these two philosophies to produce a creative philosophy of mathematics rooted in the objectivity of physical reality and yet open to the astounding creativity that characterizes the human condition? I believe this is the direction the philosophy of mathematics should pursue.

Finite mathematics is objective because we can physically build at least some of what it talks about. Among the finite objects we can construct are precise sets of rules in the form of computer programs. One element common to all approaches to the philosophy of mathematics described here can be made, through technology, to approach the absolute perfection of Platonic philosophy. The execution of computer programs, in contrast to semi-formal mathematical proofs, obey a rigorous set of rules (defined by the characteristics of the machine they are running on) with something approaching absolute certainty. We cannot construct a perfect circle but we can compute the ratio of the circumference to the diameter of the perfect circle, π , to a million or more decimal places with a very high certainty that we have done it correctly. The same is true for the diagonal of the perfect square. We can write a program that could, if it were possible for it to run forever with no errors, eventually output each digit of π or $\sqrt{2}$.

There is a basis in physical reality for the perfection (or something very close to it) that Plato first described. However, when we move beyond finite questions and procedures, things become more ambiguous. This first happens in mathematics when we ask if some recursive property is true of all integers. To be recursive the condition must be verifiable in a finite number of finite operations for each integer. The ability to verify the condition for *each* integer does not allow one to determine the question for *all* integers. Such questions are cultural creations. There is no physical reality that embodies the solution. Yet such questions are logically determined by a recursively enumerable set of events i. e. by a set of events that can be output by a single computer program that runs error free forever and has access to unlimited storage.

One can of course argue that the universe is not infinite or potentially infinite and thus such questions do not have a connection with our physical reality. Brian Rotman, a social constructivist, has written a book objecting to the idea of potential infinity[5]. We can never know if the universe is potentially infinite, but it would be hard to prove this is not the case. Throughout history, theories of the universe have given a limit to its size. Those limits have repeatedly been vastly expanded. Cosmology is, of necessity, a highly speculative science. It projects what we think we understand about physical reality over vast distances and epochs of time using very limited information. The existence of *ultimate* limits to the size of the universe will be an open question for the foreseeable future even as the dominant cosmology confidently quotes its estimate of the size of the universe.

Because of this uncertainty and more importantly because of the practical

value of proofs about properties of all integers, I assume such questions are meaningful and objectively true or false even though there exists no general method for deciding them. This is where I part company with both constructivism and social constructivism. On the other hand I do not come remotely close to embracing the hierarchy of infinity in the Platonic philosophy of mathematics. For me infinity is deeply connected to the creative evolution over time that characterizes biological evolution and is the richest and most interesting aspect of existence that I know of.

Objective Mathematics

Questions about all integers have a unique existential status with their tenuous connection to physical reality. On the one hand they are logically determined by a recursively enumerable sequence of events. They can be falsified by counter examples. However there is no general way to determine if they are true. Gödel's Incompleteness Theorem established this. Gödel proved that any system that is powerful enough to embed any computer program (or equivalently embed the primitive recursive functions) must be incomplete. In particular such formal systems could not prove their own consistency unless they were inconsistent.

The consistency of any formal system is equivalent to the halting problem for some particular computer program. (The halting problem asks will a computer program, with access to unlimited storage and able to run error free forever, eventually halt.) Similarly all integers satisfy some recursive property if and only if some particular computer program never halts. In the former case we can program the computer to enumerate every theorem of the formal system and test each theorem against all previous theorems. If a contradiction is discovered the program halts. Otherwise it runs forever. In the latter case we program the computer to test the condition for each integer in succession and halt if and only if the condition fails for some integer.

The question does a computer have an infinite number of outputs has an even more tenuous connection to objective reality than does the halting problem. The latter question can be falsified by a finite event, the former cannot. To prove a program has an infinite number of outputs requires some form of induction. The proof can be trivial. A program might produce an output and then return to exactly the same state it had before the output was generated implying it will loop forever continually producing new outputs. But no finite event can establish or contradict this proof.

How far do we take this process? What is objective mathematics? It turns out that much of set theory is about questions that are determined by a recursively enumerable sequence of events. The arithmetical hierarchy includes all questions defined by a recursive relationship and a finite number of existential and universal quantifiers over the integers. This hierarchy is equivalent to the series of questions does a computer have an infinite number of outputs, does it have an infinite number of outputs such that an infinite

subset of these are programs that themselves have an infinite number of outputs etc. The n th question in this hierarchy asks does the computer have an infinite number of outputs and infinite subset of which satisfy the $n - 1$ condition in the hierarchy. A single computer program can nondeterministically³ enumerate all events at any level in this hierarchy by simulating what all computer programs do for all time. To enumerate all computer programs take the programming language for any Universal Turing Machine⁴ and generate all possible finite sequences in that language. Then nondeterministically simulate each of these programs to enumerate what every computer program will do at every point in time.

Even some questions that require quantification over the reals are determined by a recursively enumerable sequence of events. For example consider a computer program that accepts an integer as input and outputs either 0 or another computer program with the same definition as itself. We can apply a sequence of integers to such a program. The first integer is applied to the base program. If we get 0 we do nothing and say the path terminates. Otherwise we apply the next integer in the sequence to the computer program that was previously output. We keep doing this for every integer in the sequence until and unless we get a zero output. If applying every sequence of integers to the base computer results in a terminated path after some finite number of steps the original base computer program is said to be well founded.

Asking if a computer program is well founded requires quantification over the reals. Every question in the hyperarithmetical hierarchy is equivalent to the question is a particular computer program well founded. Yet each such question is determined by a recursively enumerable sequence of events. A single program can nondeterministically do what every computer program will do for every integer input. This includes all the events that determine if a computer program is well founded.

The property of a computer program being well founded is impredicative. Properties defined in terms of all reals or all infinite sequences of integers can be circular because they can be used to define new reals. This is an issue because such circular definitions have, in the past, led to contradictions in mathematical systems. Claiming that some impredicative definitions must be objectively true or false crosses a major boundary. None the less I think a computer being well founded is an objective property with important practical significance.

We can attribute a limited form of objectivity to any question logically determined by a recursively enumerable sequence of events. Everything that determines the outcome could happen in a finite but potentially infinite

³A nondeterministic computer program can simulate an infinite sequence of computer programs by simulating the first program for one time step, followed by simulating the first two programs for two time steps, followed by simulating the first three programs for three time steps, etc.

⁴ A Universal Turing Machine is a computer that can simulate any computer that it is possible to build. It can be a simple device, but it requires access to unlimited storage.

universe. This definition separates the Continuum Hypothesis (not objectively true or false) from the question of whether a computer program is well founded. However it does not constrain objective mathematics to a particular set of propositions.

A Creative Objective Philosophy

A philosophy of mathematics must deal with two opposing forces. Computer technology allows us to create, in limited ways, structures that can approach the ideal perfection of Plato's philosophy. One can never eliminate all possibility of error but, in limited domains and with enough resources, the error rate can be made arbitrarily small. Today's computers perform billions of operations a second with rare hardware failures. Application and even operating system program bugs are far more common but the basic hardware is extremely stable.

Simple programs carefully reviewed can be error free. Complex programs are another matter. However *what they produce* can be made relatively error free. The largest computer chips today have hundreds of millions of switches and can only be designed with the aid of computer programs. Those programs are not error free but the entire design process allows one to produce a chip that ultimately is error free. Furthermore one must be able to detect all manufacturing faults in every chip produced. Thus the computer chip must be designed to make such verification possible. A limited form of Plato's heaven exists today in the engineering labs of Intel and AMD.

The opposing force is Gödel's Incompleteness Theorem and its implications. The hope that there can be a precise set of rules that determine all mathematical truth has been dashed forever. There can be no general solution even to a question as basic as the halting problems for computers. For me this is a reflection of the creative reality of our existence. One cannot determine all mathematical truth, even in a potentially infinite universe, but one can *explore* all of it in such a universe. If we insist on a single approach to mathematics we will inevitably run up against a Gödelian limit. This will not be a fixed limit or specific event. Rather it will be never ending progress that continually generates new results. However the collection of all these results will be subsumed in a single mathematical truth that we will never discover or explore.

If, on the other hand, mathematics becomes a divergent process that continually explores ever more possibilities, then there is no limit to the mathematics we may explore. This may seem as fanciful as Plato's heaven or a measurable cardinal but a divergent process, biological evolution, created the mathematically capable human mind. Evolution on this planet is enormously diverse. Over a vast expanse of time this diversity has increased enormously from the first reproducing molecules to today's biosphere. It is a safe bet that without this diversity, the enormous complexity and enormous depth of the human mind could never have evolved.

This suggests to me that the stakes are much higher than what happens with our mathematical knowledge. The hierarchy of mathematical truth involves ever more complex levels of abstraction and self reflection. The evolution of the mathematically capable human mind and the evolution of the depth and richness of human consciousness both seem to depend in part on the rich and subtle powers of abstraction and self reflection that uniquely characterize human thought and awareness. We are entering a unique period in biological evolution. Evolution has become, through us, conscious of itself and is acquiring the tools to control its future destiny. Understanding the role of diversity in not limiting the potential of future evolution may be crucial to what we can become [2].

A Cultural Prescription

Ironically the key to expanding mathematical diversity lies in embracing the technology through which humanity has obtained something approaching Platonic perfection. One must turn the foundations of mathematics into an experimental science embracing computer technology as an essential research tool just as every other major branch of science has done.

There is a cultural bias in mathematics to come up with the simplest most elegant approach possible. Most mathematical research is done using pencil, paper and the mathematician's mind, limiting the complexity that can be dealt with. Computers may be used to replace pencil and paper but they are rarely used as a research tool or to verify proofs. Of course elegance and simplicity are worthy goals, but one must not insist on them to the point of failing to deal with the enormous complexity that the foundations of mathematics suggests we can explore. The strength of a formal system is determined, in large measure, by the ordinals definable within it. Notations for recursive ordinals and recursive operations on these notation can be explored experimentally using computers. Recent history of science suggests that leveraging human intuition with the combinatorial power of computers will lead to results far beyond what the unaided human mind is capable of. I do not think the foundations of mathematics will be an exception.

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